

An improved approximation to l -wave bound states of the Manning-Rosen potential by Nikiforov-Uvarov method

Sameer M. Ikhdaire and Ramazan Sever¹

^{1*}*Department of Physics, Near East University, Nicosia, Cyprus, Turkey.*

[†]*Department of Physics, Middle East Technical University, 06531 Ankara, Turkey.*

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Abstract

A new approximation scheme to the centrifugal term is proposed to obtain the $l \neq 0$ solutions of the Schrödinger equation with the Manning-Rosen potential. We also find the corresponding normalized wave functions in terms of the Jacobi polynomials. To show the accuracy of the new approximation scheme, we calculate the energy eigenvalues numerically for arbitrary quantum numbers n and l with two different values of the potential parameter α . The bound state energies of various states for a few HCl , CH , LiH and CO diatomic molecules are also calculated. The numerical results are in good agreement with those obtained by using program based on a numerical integration procedure. Our solution can be also reduced to the s -wave ($l = 0$) case and to the Hulthén potential case.

Keywords: Bound states; Manning-Rosen potential; Nikiforov-Uvarov method.

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I. INTRODUCTION

The exact analytic solutions of the wave equations (nonrelativistic and relativistic) are only possible for certain potentials of physical interest under consideration since they contain all the necessary information on the quantum system. It is well known that the exact solutions of these wave equations are only possible in a few simple cases such as the Coulomb, the harmonic oscillator, pseudoharmonic potentials and others [1-5]. Recently, the analytic exact solutions of the wave equation with some exponential-type potentials are impossible for $l \neq 0$ states. Approximation methods have to be used to deal with the centrifugal term like the Pekeris approximation [6-8] and the approximated scheme suggested by Greene and Aldrich [9]. Some of these exponential-type potentials include the Morse potential [10], the Hulthén potential [11], the Pöschl-Teller [12], the Woods-Saxon potential [13], the Kratzer-type and pseudoharmonic potentials [14], the Rosen-Morse-type potentials [15], the Manning-Rosen potential [15-19] and other multiparameter exponential-type potentials [20,21].

The Manning-Rosen (M-R) potential has been one of most useful and convenient models to study the energy eigenvalues of diatomic molecules [16]. As an empirical potential, the M-R potential gives an excellent description of the interaction between the two atoms in a diatomic molecule and also it is very reasonable to describing the interactions close to the surface. The short range M-R potential is defined by [15-19]

$$V(r) = -\frac{A\hbar^2}{2\mu b^2} \frac{e^{-r/b}}{1 - e^{-r/b}} + \frac{\alpha(\alpha - 1)\hbar^2}{2\mu b^2} \left(\frac{e^{-r/b}}{1 - e^{-r/b}} \right)^2, \quad (1)$$

where A and α are two-dimensionless parameters [22] but the screening parameter b has dimension of length which has a potential range $1/b$. The potential (1) may be further put in the following simple form

$$V(r) = -\frac{Ce^{-r/b} + De^{-2r/b}}{(1 - e^{-r/b})^2}, \quad C = A, \quad D = -A - \alpha(\alpha - 1), \quad (2)$$

which is usually used for the description of diatomic molecular vibrations [23,24]. It is also used in several branches of physics for their bound states and scattering properties. The potential in (1) remains invariant by mapping $\alpha \rightarrow 1 - \alpha$ and has a relative minimum value $V(r_0) = -\frac{A^2}{4\kappa b^2 \alpha(\alpha-1)}$ at $r_0 = b \ln \left[1 + \frac{2\alpha(\alpha-1)}{A} \right]$ for $\alpha > 0$ to be obtained from the first derivative $\left. \frac{dV}{dr} \right|_{r=r_0} = 0$. The second derivative which determines the force constants at $r = r_0$ is given by

$$\left. \frac{d^2 V}{dr^2} \right|_{r=r_0} = \frac{A^2 [A + 2\alpha(\alpha - 1)]^2}{8b^4 \alpha^3 (\alpha - 1)^3}. \quad (3)$$

It is known that for this potential the Schrödinger equation (SE) can be solved for the s -wave, angular momentum quantum number $l = 0$. However, in general solution, it is needed to include some approximations if one wants to obtain analytical or semianalytical solutions to the SE. Also, it is often necessary to determine the l -wave ($l \neq 0$ states), so an analytic procedure would be advantageous [25-27]. Hence, in the previous papers, several approximations have been developed to find better analytical formulas for the M-R potential. For instance, in the $l = 0$ case, the bound-state energy eigenvalues for the M-R potential have already been calculated by using the path-integral approach [17] and function analysis method [18]. For the $l \neq 0$ case, the potential can not be solved exactly without approximation. Recently, Qiang and Dong [19] approximated the centrifugal term

$$\frac{1}{r^2} \approx \frac{1}{b^2} \left[\frac{e^{-r/b}}{1 - e^{-r/b}} + \left(\frac{e^{-r/b}}{1 - e^{-r/b}} \right)^2 \right] = \frac{1}{b^2} \frac{e^{-r/b}}{(1 - e^{-r/b})^2}$$

and studied l -wave bound state solutions of the SE with M-R potential. Wei *et al.* [25] investigated the scattering state solutions of the SE with M-R potential using the approximation [9,11,19]. Ikhdair and Sever [11,26,27] applied the above approximation and obtained the l -wave solutions of SE with the M-R potential in three-dimensions and D -dimensions and also with the Hulthén potential using Nikiforov and Uvarov (N-U) method. These approximations provide good results which are in agreement with the numerical integration method by Lucha and Schöberl [28] for short-range potential (large b and small l) but not for long-range potential (small b and large l).

Our aim is to improve the accuracy of our previous approximation [26,27], so that we propose and apply a new approximation scheme for the centrifugal term to get our results in high agreement with Ref. [28]. Thus, with this new approximation scheme, we calculate the $l \neq 0$ energy levels and wavefunctions of the M-R potential using the Nikiforov and Uvarov (N-U) method which has shown its power in calculating the exact energy levels for some solvable quantum systems. For this, the results are in better agreement with those obtained by means of numerical integration method [28]. As an illustration, the method is applied to find the energy levels of the HCl , LiH , CH and CO diatomic molecules.

The paper is organized as follows: In Section II we briefly present the Nikiforov-Uvarov (N-U) method. In Section III, we present the new proposed approximation scheme and

apply it to calculate the l -wave bound state eigensolutions of the SE with M-R potential by the N-U method. In Section IV, we present our numerical results for various diatomic molecules. Section V, is devoted to for two special cases, namely, s -wave ($l = 0$) and the Hulthén potential. Finally, we make a few concluding remarks in Section VI.

II. THE NIKIFOROV AND UVAROV METHOD

The N-U method is based on solving the second-order linear differential equation by reducing it to a generalized equation of hypergeometric type [29]. In this method after employing an appropriate coordinate transformation $z = z(r)$, the Schrödinger equation can be written in the following form:

$$\psi_n''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi_n'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi_n(z) = 0, \quad (4)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are the polynomials with at most of second-degree, and $\tilde{\tau}(z)$ is a first-degree polynomial. The special orthogonal polynomials [29] reduce Eq. (4) to a simple form by employing $\psi_n(z) = \phi_n(z)y_n(z)$, and choosing an appropriate function $\phi_n(z)$. Consequently, Eq. (4) can be reduced into an equation of the following hypergeometric type:

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0, \quad (5)$$

where $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ (its derivative must be negative) and λ is a constant given in the form

$$\lambda = \lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z), \quad n = 0, 1, 2, \dots \quad (6)$$

It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(z) = y_n(z)$ which is a polynomial of degree n . Further, $y_n(z)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (7)$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition [29]

$$\frac{d}{dz}w(z) = \frac{\tau(z)}{\sigma(z)}w(z), \quad w(z) = \sigma(z)\rho(z). \quad (8)$$

In order to determine the weight function given in Eq. (8), we must obtain the following polynomial:

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}. \quad (9)$$

In principle, the expression under the square root sign in Eq. (9) can be arranged as the square of a polynomial. This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are included in the N-U method and here there is a relationship between λ and k by $k = \lambda - \pi'(z)$. After this point an appropriate $\phi_n(z)$ can be extracted from the condition

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}. \quad (10)$$

III. ANALYTICAL SOLUTIONS

A. An Improved Approximation Scheme

The approximation is based on the expansion of the centrifugal term in a series of exponentials depending on the intermolecular distance r and keeping terms up to second order. Therefore, instead of using the approximation in [9,11,19], we use this choice of approximation:

$$\begin{aligned} \frac{1}{r^2} &\approx \frac{1}{r_0^2} [c_0 + c_1 v(r) + c_2 v^2(r)], \quad v(r) = \frac{e^{-r/b}}{1 - e^{-r/b}} \\ \frac{1}{r^2} &\approx \frac{1}{r_0^2} \left[c_0 + c_1 \frac{1}{e^{r/b} - 1} + c_2 \frac{1}{(e^{r/b} - 1)^2} \right], \end{aligned} \quad (11)$$

which has a similar form of the M-R potential. Changing the coordinate to x by using $x = (r - r_0)/r_0$, one obtains

$$(1+x)^{-2} = \left[c_0 + \frac{c_1}{e^{\gamma(1+x)} - 1} + \frac{c_2}{(e^{\gamma(1+x)} - 1)^2} \right], \quad \gamma = r_0/b. \quad (12)$$

and *expanding* Eq. (12) around $r = r_0$ ($x = 0$), we obtain the following Taylor's expansion:

$$(1 - 2x + \dots) = \left[\left(c_0 + \frac{c_1}{e^\gamma - 1} + \frac{c_2}{(e^\gamma - 1)^2} \right) - \gamma \left(\frac{c_1}{e^\gamma - 1} + \frac{c_1 + 2c_2}{(e^\gamma - 1)^2} + \frac{2c_2}{(e^\gamma - 1)^3} \right) x + \dots \right], \quad (13)$$

from which we obtain

$$\begin{aligned} c_0 + \frac{c_1}{e^\gamma - 1} + \frac{c_2}{(e^\gamma - 1)^2} &= 1, \\ \gamma \left(\frac{c_1}{e^\gamma - 1} + \frac{c_1 + 2c_2}{(e^\gamma - 1)^2} + \frac{2c_2}{(e^\gamma - 1)^3} \right) &= 2. \end{aligned} \quad (14)$$

Taking $r_0 = b$ ($\gamma = 1$), one obtains, from Eq. (14), the following three simple cases:

Case 1. If $c_1 = c_2 = 1$, then the shift, c_0 , in the present approximation is simply given by

$$c_0 = 1 - \frac{1}{e - 1} - \frac{1}{(e - 1)^2} = 0.0793264057923, \quad (15)$$

where e is the base of the natural logarithms, $e = 2.718281828459045$.

Case 2. Without any loss of generality, we may take $c_1 = 1$, then we can calculate the shift $c_0 = 0.0768910877367$ and $c_2 = 1.007190258153$.

Case 3. If we choose $c_2 = 1$, then we find the shift $c_0 = 0.0744557696812$ and $c_1 = 1.0083691255228$. Thus, for the approximation given in case 1, we have.

$$\lim_{b \rightarrow \infty} \frac{1}{b^2} \left[1 - \frac{1}{e - 1} - \frac{1}{(e - 1)^2} + \frac{e^{-r/b}}{1 - e^{-r/b}} + \left(\frac{e^{-r/b}}{1 - e^{-r/b}} \right)^2 \right] = \frac{1}{r^2}. \quad (16)$$

Finally, in the case if $c_0 = 0$ and $c_1 = c_2 = 1$, the approximation given in Eq. (11) is identical to the commonly used approximation in the previous works [9,11,19,26,27].

B. Bound State Solutions

To study any quantum physical system characterized by the empirical potential given in Eq. (1), we solve the original SE which is given in the well known textbooks [1,2]

$$\left(\frac{p^2}{2m} + V(r) \right) \psi(\mathbf{r}, \theta, \phi) = E \psi(\mathbf{r}, \theta, \phi), \quad (17)$$

where the potential $V(r)$ is taken as the M-R form in (1). Using the separation method with the wavefunction $\psi(\mathbf{r}, \theta, \phi) = r^{-1} R(r) Y_{lm}(\theta, \phi)$, we obtain the following radial Schrödinger equation as

$$\frac{d^2 R_{nl}(r)}{dr^2} + \left[\frac{2\mu E_{nl}}{\hbar^2} - V_{eff}(r) \right] R_{nl}(r) = 0,$$

$$V_{eff}(r) = \frac{1}{b^2} \left[\frac{\alpha(\alpha-1)e^{-2r/b}}{(1-e^{-r/b})^2} - \frac{Ae^{-r/b}}{1-e^{-r/b}} \right] + \frac{l(l+1)}{r^2}. \quad (18)$$

Since the SE with the above M-R effective potential has no analytical solution for l -waves, the approximation to the centrifugal term given by case 1 has to be made so that the energy eigenvalues are found to be in better agreement with those obtained by means of the numerical integration method [28]. The other approximations will be left for future investigations. To solve it by the N-U method, we need to recast Eq. (18) with Eq. (16) into the form of Eq. (4) changing the variables $r \rightarrow z$ through the mapping function $r = f(z)$ and energy transformation given by

$$z = e^{-r/b}, \quad \varepsilon' = \sqrt{-\frac{2\mu b^2 E_{nl}}{\hbar^2} + \Delta E_l}, \quad E_{nl} < \frac{\hbar^2}{2\mu b^2} \Delta E_l, \quad \Delta E_l = l(l+1)c_0, \quad (19)$$

to obtain the following hypergeometric equation:

$$\frac{d^2 R(z)}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{dR(z)}{dz} + \frac{1}{[z(1-z)]^2} \{ -\varepsilon'^2 + [A + 2\varepsilon'^2 - l(l+1)]z - [A + \varepsilon'^2 + \alpha(\alpha-1)]z^2 \} R(z) = 0. \quad (20)$$

We notice that for bound state (real) solutions, the last equation requires that

$$z = \begin{cases} 0, & \text{when } r \rightarrow \infty, \\ 1, & \text{when } r \rightarrow 0, \end{cases} \quad (21)$$

and thus the finite radial wavefunctions $R_{nl}(z) \rightarrow 0$. To apply the N-U method, we compare Eq. (20) with Eq. (4) and obtain the following values for the parameters:

$$\tilde{\tau}(z) = 1-z, \quad \sigma(z) = z-z^2, \quad \tilde{\sigma}(z) = -[A + \varepsilon'^2 + \alpha(\alpha-1)]z^2 + [A + 2\varepsilon'^2 - l(l+1)]z - \varepsilon'^2. \quad (22)$$

If one inserts these values of parameters into Eq. (9), with $\sigma'(z) = 1-2z$, the following linear function is achieved

$$\pi(z) = -\frac{z}{2} \pm \frac{1}{2} \sqrt{\{1 + 4[A + \varepsilon'^2 + \alpha(\alpha-1)] - k\}z^2 + 4\{k - [A + 2\varepsilon'^2 - l(l+1)]\}z + 4\varepsilon'^2}. \quad (23)$$

According to this method the expression in the square root has to be set equal to zero, that is, $\Delta = \{1 + 4[A + \varepsilon'^2 + \alpha(\alpha - 1)] - k\} z^2 + 4\{k - [A + 2\varepsilon'^2 - l(l + 1)]\} z + 4\varepsilon'^2 = 0$. Thus the constant k can be determined as

$$k = A - l(l + 1) \pm a\varepsilon', \quad a = \sqrt{(1 - 2\alpha)^2 + 4l(l + 1)}. \quad (24)$$

In this regard, we can find four possible functions for $\pi(z)$ as

$$\pi(z) = -\frac{z}{2} \pm \begin{cases} \varepsilon - (\varepsilon' - \frac{a}{2}) z, & \text{for } k = A - l(l + 1) + a\varepsilon', \\ \varepsilon - (\varepsilon' + \frac{a}{2}) z; & \text{for } k = A - l(l + 1) - a\varepsilon'. \end{cases} \quad (25)$$

We must select

$$k = A - l(l + 1) - a\varepsilon', \quad \pi(z) = -\frac{z}{2} + \varepsilon' - \left(\varepsilon + \frac{a}{2}\right) z, \quad (26)$$

in order to obtain the polynomial, $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ having negative derivative as

$$\tau(z) = 1 + 2\varepsilon' - (2 + 2\varepsilon' + a) z, \quad \tau'(z) = -(2 + 2\varepsilon' + a). \quad (27)$$

We can also write the values of $\lambda = k + \pi'(z)$ and $\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z)$, $n = 0, 1, 2, \dots$ as

$$\lambda = A - l(l + 1) - (1 + a) \left[\frac{1}{2} + \varepsilon' \right], \quad (28)$$

$$\lambda_n = n(1 + n + a + 2\varepsilon'), \quad n = 0, 1, 2, \dots \quad (29)$$

respectively. Additionally, using the definition of $\lambda = \lambda_n$ and solving the resulting equation for ε' , allows one to obtain

$$\varepsilon' = \frac{(n + 1)^2 + l(l + 1) + (2n + 1)\Lambda - A}{2(n + 1 + \Lambda)}, \quad \Lambda = \frac{-1 + a}{2}. \quad (30)$$

Using Eqs. (19) and (30), we obtain the discrete energy levels

$$E_{nl} = -\frac{\hbar^2}{2\mu b^2} \left[\frac{(n + 1)^2 + l(l + 1) + (2n + 1)\Lambda - A}{2(n + 1 + \Lambda)} \right]^2 + \frac{\hbar^2 l(l + 1) c_0}{2\mu b^2}, \quad 0 \leq n, l < \infty, \quad (31)$$

where $n = 0, 1, 2, \dots$ and l signify the usual radial and angular momentum quantum numbers, respectively. It is found that Λ remains invariant by mapping $\alpha \rightarrow 1 - \alpha$, so do the

bound state energies E_{nl} . An important quantity of interest for the M-R potential is the critical coupling constant A_c , which is that value of A for which the binding energy of the level in question becomes zero. Hence, using Eq. (31), in atomic units $\hbar^2 = \mu = Z = e = 1$, we find the following critical coupling constant

$$A_c = \left(n + 1 + \Lambda - \sqrt{l(l+1)d_0} \right)^2 - \Lambda(\Lambda+1) + l(l+1)(1-d_0). \quad (32)$$

Let us now find the corresponding radial part of the normalized wave functions. Using $\sigma(z)$ and $\pi(z)$ in Eqs. (22) and (26), we obtain

$$\phi(z) = z^{\varepsilon'} (1-z)^{(\Lambda+1)/2}, \quad (33)$$

$$\rho(z) = z^{2\varepsilon'} (1-z)^{2\Lambda+1}, \quad (34)$$

$$y_{nl}(z) = C_n z^{-2\varepsilon'} (1-z)^{-(2\Lambda+1)} \frac{d^n}{dz^n} \left[z^{n+2\varepsilon'} (1-z)^{n+2\Lambda+1} \right]. \quad (35)$$

The functions $y_{nl}(z)$, up to a numerical factor, are in the form of Jacobi polynomials, i.e., $y_{nl}(z) \simeq P_n^{(2\varepsilon', 2\Lambda+1)}(1-2z)$, valid physically in the interval $(0 \leq r < \infty \rightarrow 0 \leq z \leq 1)$ [30]. Therefore, the radial part of the wave functions can be found by substituting Eqs. (33) and (35) into $R_{nl}(z) = \phi(z)y_{nl}(z)$ as

$$R_{nl}(z) = N_{nl} z^{\varepsilon'} (1-z)^{1+\Lambda} P_n^{(2\varepsilon', 2\Lambda+1)}(1-2z), \quad (36)$$

where ε and Λ are given in Eqs. (24) and (30) and N_{nl} is a normalization factor to be determined from the normalization condition:

$$\int_0^\infty |R_{nl}(r)|^2 dr = 1 = b \int_0^1 z^{-1} |R_{nl}(z)|^2 dz. \quad (37)$$

This can be further written as

$$1 = b N_{nl}^2 \int_0^1 z^{2\varepsilon'-1} (1-z)^{2\Lambda+2} \left[P_n^{(2\varepsilon', 2\Lambda+1)}(1-2z) \right]^2 dz. \quad (38)$$

from which we obtain [26]

$$N_{nl} = \frac{1}{\sqrt{s(n)}}, \quad (39)$$

$$s(n) = b(-1)^n \frac{\Gamma(n + 2\Lambda + 2)\Gamma(n + 2\varepsilon' + 1)^2}{\Gamma(n + 2\varepsilon' + 2\Lambda + 2)} \\ \times \sum_{p,r=0}^n \frac{(-1)^{p+r}\Gamma(n + 2\varepsilon' + r - p + 1)(p + 2\Lambda + 2)}{p!r!(n - p)!(n - r)!\Gamma(n + 2\varepsilon' - p + 1)\Gamma(2\varepsilon' + r + 1)(n + 2\varepsilon' + r + 2\Lambda + 2)}. \quad (40)$$

IV. NUMERICAL RESULTS

To show the accuracy of the new approximation scheme, we have calculated the energy eigenvalues for various n and l quantum numbers with two different values of the parameters α . The results calculated by Eq. (31) are compared with those obtained by a MATHEMATICA package programmed by Lucha and Schöberl [28] as shown in Table 1 for short-range (large b) and long-range (small b) potentials. This is an illustration to assess the validity and usefulness of our present calculation. The energy eigenvalues for a few HCl , CH , LiH and CO diatomic molecules are presented in Tables 2 and 3. Lowest eigenvalues of $l = 0, 1, 2, 3$ are given at four values of $1/b$ in the range $0.025 - 0.1$ covering both weaker and stronger interaction to demonstrate the generality of our results. The formalism is quite simple, computationally efficient, reliable and illustrated very accurate.

V. DISCUSSIONS

In this work, we have utilized N-U method and solved the radial SE for the M-R model potential with the angular momentum $l \neq 0$ states. We have derived the binding energy spectra in Eq. (31) and their corresponding wave functions in Eq. (36).

Let us study special cases. We have shown that for $\alpha = 0$ (1), the present solution reduces to the one of the Hulthén potential [9,11]:

$$V^{(H)}(r) = -V_0 \frac{e^{-\delta r}}{1 - e^{-\delta r}}, \quad V_0 = Ze^2\delta, \quad \delta = b^{-1} \quad (41)$$

where Ze^2 is the strength and δ is the screening parameter and b is the range of potential. If the potential is used for atoms, the Z is identified with the atomic number. This can be achieved by setting $\Lambda = l$, hence, the energy for $l \neq 0$ states

$$E_{nl} = -\frac{[A - (n + l + 1)^2]^2 \hbar^2}{8\mu b^2 (n + l + 1)^2} + \frac{\hbar^2 l(l + 1)c_0}{2\mu b^2}, \quad 0 \leq n, l < \infty. \quad (42)$$

and for s -wave ($l = 0$) states

$$E_n = -\frac{[A - (n+1)^2]^2 \hbar^2}{8\mu b^2 (n+1)^2}, \quad 0 \leq n < \infty \quad (43)$$

Essentially, these results coincide with those obtained by the Feynman integral method [17] and the standard way [18,19], respectively. Furthermore, if taking $b = 1/\delta$ and identifying $\frac{A\hbar^2}{2\mu b^2}$ as $Ze^2\delta$, we are able to obtain

$$E_{nl} = -\frac{\mu(Ze^2)^2}{2\hbar^2} \left[\frac{1}{n+l+1} - \frac{\hbar^2\delta}{2Ze^2\mu}(n+l+1) \right]^2 + \frac{\hbar^2 l(l+1)c_0\delta^2}{2\mu}, \quad (44)$$

and using the natural units $\hbar^2 = \mu = Z = e = 1$, we further obtain

$$E_{nl} = -\frac{1}{2} \left[\frac{1}{n+l+1} - \frac{(n+l+1)}{2} \delta \right]^2 + \frac{l(l+1)c_0\delta^2}{2}. \quad (45)$$

The corresponding radial wave functions are expressed as

$$R_{nl}(r) = N_{nl} e^{-\delta\varepsilon' r} (1 - e^{-\delta r})^{l+1} P_n^{(2\varepsilon', 2l+1)}(1 - 2e^{-\delta r}), \quad (46)$$

where

$$\varepsilon' = \frac{\mu Ze^2}{\hbar^2\delta} \left[\frac{1}{n+l+1} - \frac{\hbar^2\delta}{2Ze^2\mu}(n+l+1) \right], \quad 0 \leq n, l < \infty, \quad (47)$$

which coincides for the ground state with Gönül *et al.* [9] in Eq. (6). In addition, for $\delta r \ll 1$ (i.e., $r/b \ll 1$), the Hulthén potential turns to become a Coulomb potential: $V(r) = -Ze^2/r$ with energy levels and wavefunctions:

$$E_{nl} = -\frac{\varepsilon_0}{(n+l+1)^2}, \quad n = 0, 1, 2, \dots$$

$$\varepsilon_0 = \frac{Z^2\hbar^2}{2\mu a_0^2}, \quad a_0 = \frac{\hbar^2}{\mu e^2} \quad (48)$$

where $\varepsilon_0 = 13.6$ eV and a_0 is Bohr radius for the Hydrogen atom [3]. The wave functions are

$$R_{nl} = N_{nl} \exp \left[-\frac{\mu Ze^2}{\hbar^2} \frac{r}{(n+l+1)} \right] r^{l+1} P_n^{\left(\frac{2\mu Ze^2}{\hbar^2\delta(n+l+1)}, 2l+1 \right)}(1 - 2\delta r) \quad (49)$$

which coincide with Refs. [11,13].

VI. COCLUDING REMARKS

In this work, we have used a new improved approximation to the centrifugal term and determined approximately the arbitrary l -wave bound state solution of the Schrödinger equation with the M-R potential. The special cases for $\alpha = 0, 1$ are discussed. The results are in good agreement with those obtained by other methods for short potential range, small α and l . We have also studied two special cases for $l = 0, l \neq 0$ and Hulthén potential. The results we have ended up show that the N-U method constitute a reliable alternative way in solving the exponential potentials. The numerical results show that our results are in good agreement with those obtained by using the MATHEMATICA program based on the numerical integration procedure [28]. This means that the approximation in Eq. (16) is a good approximation since the energy are very close to the ones obtained in [28]. Furthermore, we have applied this approximation in obtaining the energy bound states ($-E_{nl}$) for a few HCl , CH , LiH and CO diatomic molecules.

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TABLE I: Bound state energy eigenvalues ($-E_{nl}$) (in atomic units) for the Manning-Rosen potential as a function of $1/b$ for $2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f$ and $6g$ states with $\alpha = 0.75$, $\alpha = 1.5$ and $A = 2b$.

$\alpha = 0.75$				$\alpha = 1.5$			
states	$1/b$	present	previous [26]	Lucha et al [28]	present	previous [26]	Lucha et al [28]
$2p$	0.025	0.1205297	0.1205793	0.1205271	0.0899732	0.0900229	0.0899708
	0.050	0.1082245	0.1084228	0.1082151	0.0800489	0.0802472	0.0800400
	0.075	0.0964658	0.0969120	0.0964469	0.0705870	0.0710332	0.0705701
	0.100	0.0852807	0.0860740		0.0569224	0.0577157	
$3p$	0.025	0.0458800	0.0459297	0.0458779	0.0369154	0.0369651	0.0369134
	0.050	0.0350689	0.0352672	0.0350633	0.0272736	0.0274719	0.0272696
	0.075	0.0255647	0.0260110	0.0255654	0.0189388	0.0193850	0.0189474
	0.100	0.0173676	0.0181609		0.0119110	0.0127043	
$3d$	0.025	0.0447812	0.0449299	0.0447743	0.0394857	0.0396345	0.0394789
	0.050	0.0337133	0.0343082	0.0336930	0.0294680	0.0300629	0.0294496
	0.075	0.0237782	0.0251168	0.0237621	0.0204734	0.0218121	0.0204663
$4p$	0.025	0.0208112	0.0208608	0.0208097	0.0171753	0.0172249	0.0171740
	0.050	0.0117308	0.0119292	0.0117365	0.0089036	0.0091019	0.0089134
	0.075	0.0050311	0.0054773	0.0050945	0.0031016	0.0035478	0.0031884
$4d$	0.025	0.0203068	0.0204555	0.0203017	0.0182162	0.0183649	0.0182115
	0.050	0.0109792	0.0115742	0.0109904	0.0094998	0.0100947	0.0095167
	0.075	0.0038661	0.0052047	0.0040331	0.0029422	0.0042808	0.0031399
$4f$	0.025	0.0199911	0.0202887	0.0199797	0.0186247	0.0189223	0.0186137
	0.050	0.0102384	0.0114284	0.0102393	0.0093953	0.0105852	0.0094015
	0.075	0.0024162	0.0050935	0.0026443	0.0019754	0.0046527	0.0022307
$5p$	0.025	0.0098080	0.0098576	0.0098079	0.0080812	0.0081308	0.0080816
$5d$	0.025	0.0095150	0.0096637	0.0095141	0.0085415	0.0086902	0.0085415
$5f$	0.025	0.0092862	0.0095837	0.0092825	0.0086647	0.0089622	0.0086619
$5g$	0.025	0.0090440	0.0095398	0.0090330	0.0086252	0.0091210	0.0086150
$6p$	0.025	0.0043555	0.0044051	0.0043583	0.0034838	0.0035334	0.0034876
$6d$	0.025	0.0041574	0.0043061	0.0041650	0.0036722	0.0038209	0.0036813
$6f$	0.025	0.0039677	0.0042652	0.0039803	0.0036631	0.0039606	0.0036774
$6g$	0.025	0.0037470	0.0042428	0.0037611	0.0035464	0.0040422	0.0035623

TABLE II: Bound state energy eigenvalues ($-E_{nl}$) (in eV) for HCl and CH for $2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f$ and $6g$ states with $\hbar c = 1973.29 \text{ eV } \text{\AA}^\circ$, $\mu_{HCl} = 0.9801045 \text{ amu}$, $\mu_{CH} = 0.929931 \text{ amu}$ and $A = 2b$.

states	$1/b^a$	$HCl/$	$\alpha = 0, 1$	$\alpha = 0.75$	$\alpha = 1.5$	$CH/$	$\alpha = 0, 1$	$\alpha = 0.75$	$\alpha = 1.5$
$2p$	0.025	4.80941188	5.14067096	3.83741636	5.06889891	5.41803073	4.04446034		
	0.050	4.30992001	4.61584459	3.41413694	4.54245745	4.86488789	3.59834329		
	0.075	3.83285565	4.11432861	3.01058097	4.03965355	4.33631311	3.17301386		
	0.100	3.37821878	3.63612726	2.42777890	3.56048721	3.83231089	2.55876729		
$3p$	0.025	1.86422242	1.95681272	1.57446670	1.96480468	2.06239060	1.65941548		
	0.050	1.41471071	1.49571070	1.16323608	1.49104002	1.57641028	1.22599733		
	0.075	1.02094947	1.09035060	0.80775166	1.07603378	1.14917938	0.85133310		
	0.100	0.68293440	0.74074096	0.50801342	0.71978146	0.78070691	0.53542279		
$3d$	0.025	1.85999327	1.90994571	1.68408920	1.96034735	2.01299493	1.77495255		
	0.050	1.39779410	1.43789211	1.25682731	1.47321069	1.51547215	1.32463817		
	0.075	0.98288709	1.01415428	0.87320241	1.03591778	1.06887196	0.92031517		
	0.100	0.61526795	0.63872794	0.53322303	0.64846412	0.67318987	0.56199254		
$4p$	0.025	0.85089842	0.88761210	0.73253860	0.89680780	0.93550233	0.77206199		
	0.050	0.47136150	0.50032556	0.37974364	0.49679334	0.52732013	0.40023233		
	0.075	0.19422206	0.21457922	0.13228479	0.20470112	0.22615662	0.13942208		
$4d$	0.025	0.84666927	0.86609664	0.77693119	0.89235047	0.91282602	0.81884974		
	0.050	0.45444489	0.46826797	0.40517060	0.47896401	0.49353290	0.42703117		
	0.075	0.15615968	0.16489027	0.12548533	0.16458512	0.17378676	0.13225577		
$4f$	0.025	0.84032554	0.85263452	0.79435667	0.88566447	0.89863756	0.83721539		
	0.050	0.42906997	0.43667458	0.40071582	0.45222001	0.46023492	0.42233604		
	0.075	0.09906611	0.10305395	0.08425354	0.10441112	0.10861411	0.08879935		
$5p$	0.025	0.40106735	0.41831847	0.34466933	0.42270654	0.44088842	0.36326562		
$5d$	0.025	0.39683820	0.40581936	0.36429895	0.41824921	0.42771494	0.38395434		
$5f$	0.025	0.39049447	0.39606358	0.36955620	0.41156321	0.41743279	0.38949523		
$5g$	0.025	0.38203616	0.38573290	0.36787081	0.40264543	0.40654473	0.38771891		
$6p$	0.025	0.17707786	0.18576580	0.14858723	0.18663192	0.19578861	0.15660410		
$6d$	0.025	0.17284871	0.17731423	0.15662014	0.18217459	0.18688105	0.16507042		
$6f$	0.025	0.16650498	0.16922609	0.15623470	0.17548859	0.17835652	0.16466420		
$6g$	0.025	0.15804667	0.15981241	0.15125669	0.16657392	0.16843493	0.15941759		

TABLE III: Bound state energy eigenvalues ($-E_{nl}$) (in eV) for LiH and CO for $2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f$ and $6g$ states with $\hbar c = 1973.29 \text{ eV } \text{\AA}^\circ$, $\mu_{LiH} = 0.8801221 \text{ amu}$, $\mu_{CO} = 6.8606719 \text{ amu}$ and $A = 2b$.

states	$1/b^a$	$LiH/$	$\alpha = 0, 1$	$\alpha = 0.75$	$\alpha = 1.5$	$CO/$	$\alpha = 0, 1$	$\alpha = 0.75$	$\alpha = 1.5$
$2p$	0.025	5.35576397	5.72465427	4.27334918	1.37443170	0.73438794	0.54852071		
	0.050	4.79952952	5.14020732	3.80198495	1.23262476	0.65941210	0.48773809		
	0.075	4.26827035	4.58171881	3.35258477	1.09782989	0.58776634	0.43008673		
	0.100	3.76198647	4.04919351	2.70357604	0.97004711	0.51945126	0.34682857		
$3p$	0.025	2.07599922	2.17910783	1.75332707	0.53294169	0.27954710	0.22492577		
	0.050	1.57542270	1.66562433	1.29538040	0.40541491	0.21367481	0.16617803		
	0.075	1.13692993	1.21421508	0.89951273	0.29442115	0.15576572	0.11539410		
	0.100	0.76051617	0.82488959	0.56572406	0.19995917	0.10582106	0.07257398		
$3d$	0.025	2.07128963	2.12691670	1.87540274	0.53233752	0.27285176	0.24058626		
	0.050	1.55658435	1.60123752	1.39960364	0.40299823	0.20541494	0.17954832		
	0.075	1.09454364	1.12936281	0.97239872	0.29098362	0.14488044	0.12474428		
	0.100	0.68516276	0.71128782	0.59379748	0.19029245	0.09124764	0.07617538		
$4p$	0.025	0.94756010	0.98844538	0.81575544	0.24341803	0.12680283	0.10464928		
	0.050	0.52490845	0.55716284	0.42288275	0.13588423	0.07147570	0.05424956		
	0.075	0.21628580	0.23895554	0.14731241	0.05821126	0.03065444	0.01889799		
$4d$	0.025	0.94285141	0.96448574	0.86519105	0.24281386	0.12372917	0.11099113		
	0.050	0.50607010	0.52146349	0.45119822	0.13346755	0.06658523	0.05788202		
	0.075	0.17389951	0.18362190	0.13974054	0.05277373	0.02355596	0.01792663		
$4f$	0.025	0.93578703	0.94949432	0.88459607	0.24190761	0.12180599	0.11348051		
	0.050	0.47781258	0.48628108	0.44623738	0.12984253	0.06238263	0.05724561		
	0.075	0.11032008	0.11476093	0.09382479	0.04461744	0.01472212	0.01203632		
$5p$	0.025	0.44662885	0.46583971	0.38382398	0.11489375	0.05976030	0.04923890		
$5d$	0.025	0.44191926	0.45192068	0.40568353	0.11428958	0.05797470	0.05204316		
$5f$	0.025	0.43485488	0.44105664	0.41153801	0.11338333	0.05658100	0.05279420		
$5g$	0.025	0.42543570	0.42955239	0.40966116	0.11217500	0.05510518	0.05255343		
$6p$	0.025	0.19719402	0.20686891	0.16546683	0.05089620	0.02653820	0.02122693		
$6d$	0.025	0.19248443	0.19745724	0.17441228	0.05029203	0.02533082	0.02237450		
$6f$	0.025	0.18542005	0.18845028	0.17398306	0.04938577	0.02417537	0.02231944		
$6g$	0.025	0.17600087	0.17796720	0.16843954	0.04817743	0.02283054	0.02160829		